Interaction-free measurement

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Interaction-free measurement is introduced through a set of problems suitable for undergraduates. Both the original scheme suggested by Elitzur and Vaidman and an improved one proposed by Kwiat *et al.* are considered. Theoretical predictions are compared to experimental data.

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How often have I said to you that when you have eliminated the impossible, whatever remains, *however improbable*, must be the truth.

Arthur Conan Doyle[1]

I. INTRODUCTION

A good detective knows that much can be learned about what did occur by eliminating what could not have happened. In quantum mechanics, this idea gets turned on its head in what are known as counterfactual phenomena. Under certain circumstances, a measurement may give information about what could have happened, but did not actually occur. A good example is interaction-free measurement (IFM), which has been described as "quantum seeing in the dark." [2]

In 1993, Elitzur and Vaidman[3] proposed an experiment in which the presence of an object could sometimes be detected without absorbing a photon. They made their proposal more vivid by thinking of the object as a bomb that would explode if it absorbed just a single photon. Experiments confirming the feasibility of interaction-free measurement were carried out by Kwiat *et al.*[4] and du Marchie van Voorthuysen.[5] An improved scheme related to the quantum Zeno paradox[6] was also suggested in Ref. 4. In remarkable experiments based on this suggestion, photons signaled the presence of an object without being absorbed about three times more often than they were absorbed by the object.[7] The problems that follow provide an introduction to interaction-free measurement at a level suitable for advanced undergraduates. The theoretical solutions are compared with the published experimental results.

II. PROBLEMS

(a) The scheme for interaction-free measurement proposed in Ref. 3 is based on a Mach–Zehnder interferometer[8] which is shown in Fig. 1. It consists of two beam splitters (B1 and B2) and two mirrors (M). For simplicity, assume that the path lengths in the upper and lower arms of the interferometer are the same. First, consider the interferometer when no object blocks either arm. Show that if R_1, the reflectivity of the first beam splitter is the same as T_2, the transmittivity of the second, then a photon entering the interferometer from below will always reach the first detector (D1) and never the second (D2).

Hint: For each beam splitter, the relative phase shift between the reflected and transmitted waves is $\pi/2$, not π, because the reflected wave is actually the result of a series of internal reflections in a thin dielectric slab rather than a single reflection from a surface.[9,10]

(b) Suppose a perfectly absorbing object is placed in the lower arm of the interferometer as shown in Fig. 2. A photon may reach the first detector as before, but it also has a chance of either being absorbed by the object or reaching the second detector. The final possibility is a measurement of the object's presence without interaction, because light never reaches the second detector when the object is not present. Because a photon reaching the first detector gives no information about the presence or absence of an object, the relative probabilities of the other two possibilities should be compared to evaluate how well the apparatus is performing. Determine the efficiency of the interaction-free measurement, which is defined as the ratio of the probability of interaction-free measurement to the probability of either measurement or absorption as a function of R_1. What is the upper limit of the efficiency?

(c) In the improved arrangement proposed in Ref. 4, light is reflected through a series of beam splitters as shown in Fig. 3. (In practice, light was cycled through a pair of beam splitters several times.[7]) If the reflectivity of each beam splitter in the series is $R=\cos^2(\pi/2N)$, where N is the number of beam splitters, show that light will always reach the first detector (D1).

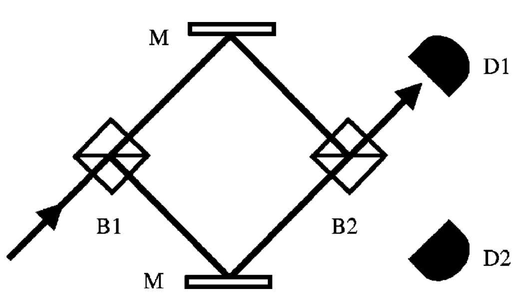

Fig. 1. Schematic of a Mach–Zehnder interferometer.

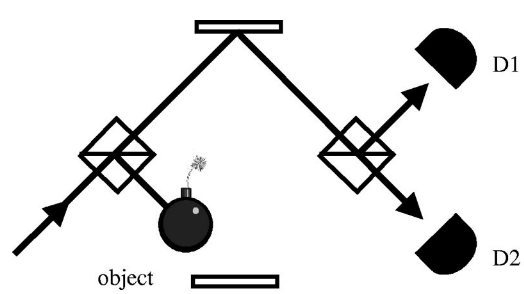

Fig. 2. The interferometer with an object in one arm.

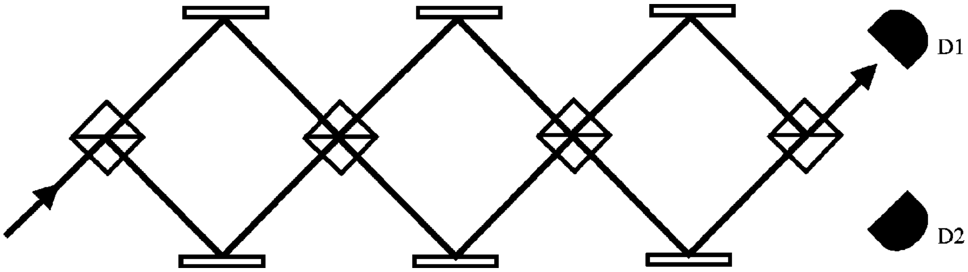

Fig. 3. The apparatus described with $N=4$.

(d) If an object blocks all of the upper arms of the apparatus in Fig. 3, a photon can also be absorbed or reach the second detector. Determine the efficiency of interaction-free measurement as a function of the number of beam splitters. What is the theoretical limit on the efficiency for this device as N becomes large?

III. SOLUTIONS

(a) In order to understand the interference that occurs as a photon leaves the system, its wave function at various locations in the apparatus will be considered. Because the probability of reflection from a beam splitter is proportional to the square of the reflected wave function, a fraction $\sqrt{R}$ of a wave function is reflected by a beam splitter. Similarly, a fraction $\sqrt{T}$ of a wave function is transmitted. In addition, there is a relative phase shift of $\pi/2$ between the reflected and transmitted waves. Let $|a\rangle$ be the wave function of a photon as it enters the interferometer. The wave function at other locations will be labeled as shown in Fig. 4. After the first beam splitter, the wave functions are

$$|b\rangle = e^{i\pi/2}\sqrt{T_1}|a\rangle = i\sqrt{T_1}|a\rangle, \tag{1}$$

and

$$|c\rangle = \sqrt{R_1}|a\rangle. \tag{2}$$

Each wave function picks up the same phase change upon reflecting from a mirror, so that this phase change can be ignored. After the second beam splitter, the wave functions emerging from the two arms interfere giving

$$|d\rangle = \sqrt{R_2}|b\rangle + i\sqrt{T_2}|c\rangle = i(\sqrt{T_1R_2} + \sqrt{R_1T_2})|a\rangle, \tag{3}$$

and

$$|e\rangle = i\sqrt{T_2}|b\rangle + \sqrt{R_2}|c\rangle = (-\sqrt{T_1T_2} + \sqrt{R_1R_2}). \tag{4}$$

Because $T_2=R_1$, it must also be true that $R_2=T_1$, because $R+T=1$ for each beam splitter. Therefore,

$$-\sqrt{T_1T_2} + \sqrt{R_1R_2} = -\sqrt{T_1R_1} + \sqrt{R_1T_1} = 0, \tag{5}$$

and $|e\rangle=0$, so the photon never reaches the second detector. The photon will always strike the first detector because that is the only other possibility, so it can also be shown that the amplitude of $|d\rangle$ is one.

(b) For this part, only the probability that a photon is going in a given direction after each beam splitter needs to be considered because no interference is occurring. If a photon is reflected by the first beam splitter, it will be absorbed, so the probability of this absorption is simply $P_{\mathrm{abs}}=R_1$. A photon that is transmitted by both beam splitters will result in an interaction-free measurement, so the probability is $P_{\mathrm{IFM}}=T_1T_2$. Therefore, the efficiency is

$$\eta = \frac{P_{\mathrm{IFM}}}{P_{\mathrm{abs}}+P_{\mathrm{IFM}}} = \frac{T_1T_2}{R_1+T_1T_2}. \tag{6}$$

Because $T_2=R_1$ and $R_1+T_1=1$, the efficiency simplifies to

$$\eta = \frac{T_1}{1+T_1} = \frac{(1-R_1)}{1+(1-R_1)} = \frac{1-R_1}{2-R_1}. \tag{7}$$

Therefore, as the reflectivity of the first beam splitter approaches zero, the efficiency approaches 1/2. As Fig. 5 shows, the theoretical efficiency agrees well with the measurements made by White *et al.*,[11] except when the reflectivity becomes very small. Unfortunately, the probability of interaction-free measurement approaches zero as the efficiency reaches its maximum, so most photons end up at the first detector and give no information. A major experimental difficulty in achieving a high efficiency and avoiding false detection of an object is the practical impossibility of obtaining total destructive interference at the second detector (D2) when no object is present in the interferometer.

A potential application of interaction-free measurement is the imaging of light-sensitive objects, such as biological systems. The preceding analysis assumes that the object is perfectly absorbing, which will not necessarily be the case. The experiments in Ref. 11 show the difficulty of imaging semi-transparent objects.[11] An analysis by Krenn, Summhammer, and Svozil suggests that the advantage of the Elitzur and Vaidman scheme[3] is lost when trying to distinguish between multiple transparencies.[12]

(c) Because $R+T=1$, the probability of transmission for each beam splitter is $T=\sin^2(\pi/2N)$. Use vectors to represent the components of the wave function in the upper and lower paths at each stage in the system. For example, a photon entering the first beam splitter from above is in the initial state $\binom{1}{0}$, and one entering from below is in the initial state $\binom{0}{1}$. The effect of a single beam splitter on a wave function in the upper path of the system is to reflect a fraction $\sqrt{R}$ back into the upper path, to transmit a fraction $\sqrt{T}$ to the

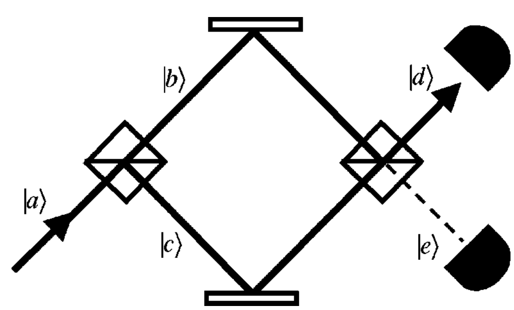

Fig. 4. The labels used for the wave functions.

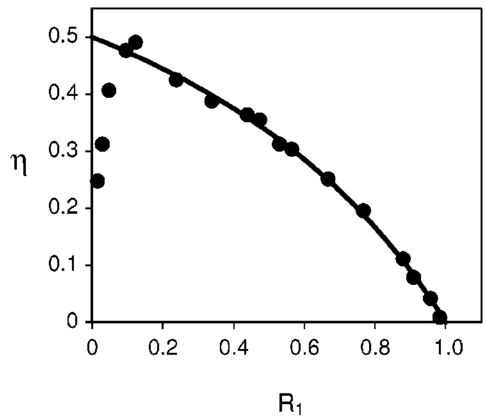

Fig. 5. The theoretical efficiency (line) compared with the experimental data (circles) for the setup suggested by Elitzur and Vaidman.

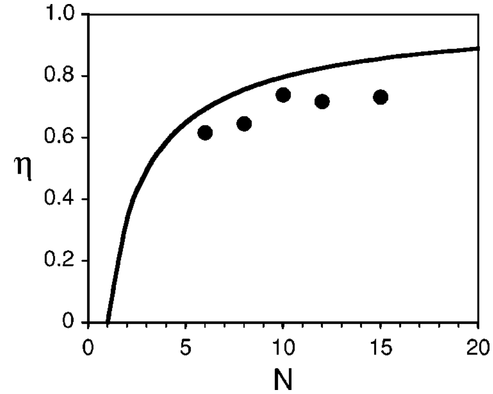

Fig. 6. The theoretical efficiency (line) compared with experimental data (circles) for the setup suggested by Kwiat *et al.*

lower path, and to introduce a relative phase shift of $\pi/2$ between the two components. This effect is represented by

$$\begin{pmatrix}1\\0\end{pmatrix}\xrightarrow{\text{beam splitter}}\begin{pmatrix}\sqrt{R}\\ i\sqrt{T}\end{pmatrix}.$$

The result for a wave function that starts completely in the lower component is just the opposite, so the evolution of a state as it passes one beam splitter is represented by the matrix operator

$$U=\begin{pmatrix}\sqrt{R} & i\sqrt{T}\\ i\sqrt{T} & \sqrt{R}\end{pmatrix}=\begin{pmatrix}\cos(\pi/2N) & i\sin(\pi/2N)\\ i\sin(\pi/2N) & \cos(\pi/2N)\end{pmatrix}. \quad (8)$$

As before, the phase changes due to the mirrors will be ignored because they are the same for the upper and lower components. If a photon enters from below as in Fig. 3, the initial state is $|\text{in}\rangle=\binom{0}{1}$, and the final state upon exiting an apparatus with N beam splitters is $|\text{out}\rangle=U^N|\text{in}\rangle$. If we define

$$\mathcal{R}(\theta)=\begin{pmatrix}\cos(\theta) & i\sin(\theta)\\ i\sin(\theta) & \cos(\theta)\end{pmatrix}, \quad (9)$$

then $U=\mathcal{R}(\pi/2N)$. The matrix $\mathcal{R}(\theta)$ is a rotation operator, which also appears in the description of spin-$\frac{1}{2}$ particles in magnetic fields.[13] The important property of rotation operators in this context is that they multiply like exponentials, that is,

$$\begin{aligned}\mathcal{R}(\theta)\mathcal{R}(\phi)&=\begin{pmatrix}\cos\theta & i\sin\theta\\ i\sin\theta & \cos\theta\end{pmatrix}\begin{pmatrix}\cos\phi & i\sin\phi\\ i\sin\phi & \cos\phi\end{pmatrix}\\ &=\begin{pmatrix}\cos(\theta+\phi) & i\sin(\theta+\phi)\\ i\sin(\theta+\phi) & \cos(\theta+\phi)\end{pmatrix}\\ &=\mathcal{R}(\theta+\phi).\end{aligned} \quad (10)$$

The evolution of a state through all N beam splitters is represented by

$$\begin{aligned}U^N&=\mathcal{R}^N(\pi/2N)\\ &=\mathcal{R}(N\cdot\pi/2N)\\ &=\begin{pmatrix}\cos(\pi/2) & i\sin(\pi/2)\\ i\sin(\pi/2) & \cos(\pi/2)\end{pmatrix}=\begin{pmatrix}0 & i\\ i & 0\end{pmatrix},\end{aligned} \quad (11)$$

so the final state is

$$|\text{out}\rangle=\begin{pmatrix}0 & i\\ i & 0\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix}=\begin{pmatrix}i\\0\end{pmatrix}. \quad (12)$$

Therefore, if a photon enters the system from below, it will leave the system from above and strike the first detector (D1). It is interesting to note that this experiment was actually performed by rotating the polarization by an angle $\theta=\pi/2N$ with a half-wave plate between successive passes through two polarizing beam splitters.[7] The analysis of that system in terms of Jones matrices is similar to the preceding one.[14]

(d) An interaction-free measurement results from reflections by all N beam splitters, so the probability is $P_{\text{IFM}}=[\cos^2(\pi/2N)]^N$. The probability of absorption following the jth beam splitter is the probability of reaching it by $(j-1)$ reflections multiplied by the probability of one transmission. Transmission by the Nth beam splitter does not result in absorption, so the total probability of absorption is

$$\begin{aligned}P_{\text{abs}}&=T+RT+R^2T+\cdots+R^{N-2}T\\ &=\sin^2\left(\frac{\pi}{2N}\right)\sum_{k=0}^{N-2}\left[\cos^2\left(\frac{\pi}{2N}\right)\right]^k.\end{aligned} \quad (13)$$

Therefore, the efficiency is

$$\eta=\frac{\cos^{2N}\left(\frac{\pi}{2N}\right)}{\cos^{2N}\left(\frac{\pi}{2N}\right)+\sin^2\left(\frac{\pi}{2N}\right)\sum_{k=0}^{N-2}\cos^{2k}\left(\frac{\pi}{2N}\right)}. \quad (14)$$

As before, this calculation makes the idealization that total destructive interference occurs at the second detector (D2) when no object is present in the apparatus. Figure 6 shows how the theoretical efficiency increases as the number of beam splitters increases, which is in reasonable agreement with the experimental data in Ref. 7. The discrepancies were explained by taking into account the imperfections of the optical components.

For large N, the probability of interaction-free measurement can be expanded in powers of $\pi/2N$ as

$$P_{\text{IFM}}=\left[1-\frac{1}{2}\left(\frac{\pi}{2N}\right)^2+\cdots\right]^{2N}. \quad (15)$$

If we keep just the first two terms inside the brackets and perform a binomial expansion, we obtain

$$P_{\mathrm{IFM}} \approx 1 - \frac{\pi^2}{4N}. \tag{16}$$

As N gets large and P_{IFM} approaches one, P_{abs} must vanish, so the efficiency approaches unity. Not only is this scheme capable of giving higher efficiencies than that of Elitzur and Vaidman,[3] but it does not suffer from the problem of the probability of interaction-free measurement becoming small as the efficiency becomes large. Jang recently used a similar system based on polarization to image semi-transparent objects with very promising results.[14]

IV. CONCLUSIONS

Interaction-free measurement is an intriguing quantum phenomenon because it defies common sense. It remains to be seen what practical applications might be developed, but there have been some interesting suggestions. For example, Vaidman has speculated that x-ray photography could be done with substantially reduced exposure,[15] but this could not be accomplished with traditional optical components. Also, a connection was recently made between interaction-free measurement and counterfactual computation, where the results can be learned from a programmed quantum computer without actually running the computer.[16] Recently, interaction-free measurement has even been brought to the attention of writers of science fiction and science fact,[17] so readers of those genres should keep an eye open for it in the future.

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