Reflections on Handedness

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When introducing the optics of mirrors, an obvious place to start is comparing the appearance of objects to their images. Image distance and magnification are usually covered the most thoroughly, but the handedness of images in various shapes of mirrors has also been discussed in *The Physics Teacher*[1–5] and elsewhere.[6] However, there has not been a systematic treatment of handedness. We describe a general criterion in terms of axis inversions for whether or not handedness is preserved by a mirror. We consider how the rotation of mirrors affects the orientation of their images. We also make some suggestions for introducing these topics in introductory courses.

Handedness and Axis Inversions

The Plane Mirror

Since the plane mirror is the most familiar, we discuss it in detail before generalizing to other mirrors. Exactly what is meant by saying that the image in a plane mirror is "reversed" has been discussed extensively.[1–3] The most elegant description is that reflection in a plane mirror reverses the direction of the axis perpendicular to it, but leaves the axes parallel to it unchanged.[1, 3] Compare looking directly at an object, as in Fig. 1(a), and looking at its reflection from a plane mirror, as in Fig. 1(b). In both cases, the rays entering the eye have the same orientation (red above, green below) so the image is not inverted in the directions parallel to the plane mirror. However, Fig. 1 also shows that if the front (white) side of an object is seen when looking directly at it, then the back (blue) side is

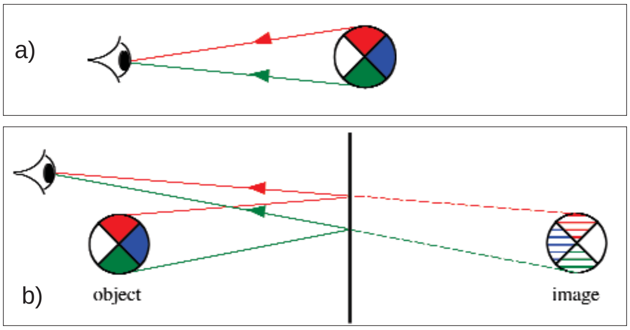

Fig. 1. (a) Viewing an object directly, and (b) viewing the image from a plane mirror.

seen when viewing the image in a plane mirror (this is not as obvious when the observer is also the object). This is what is meant by an image being inverted along the axis perpendicular to the mirror.

There is a direct relationship between axis inversions and handedness. This is obvious when the object is a right hand, with fingers defining the coordinate axes, as in Fig. 2(a). Starting with this right hand, students will find that inverting only one axis (e.g. the index finger) can only be accomplished by changing to a left hand, as in Fig. 2(b). This illustrates why the single axis inversion due to a plane mirror changes handedness.

The handedness of a coordinate system can also be treated mathematically in terms of the cross-product of its Cartesian unit vectors. For a right-handed coordinate system,

$$\hat{x} \times \hat{y} = +\hat{z}, \qquad (1)$$

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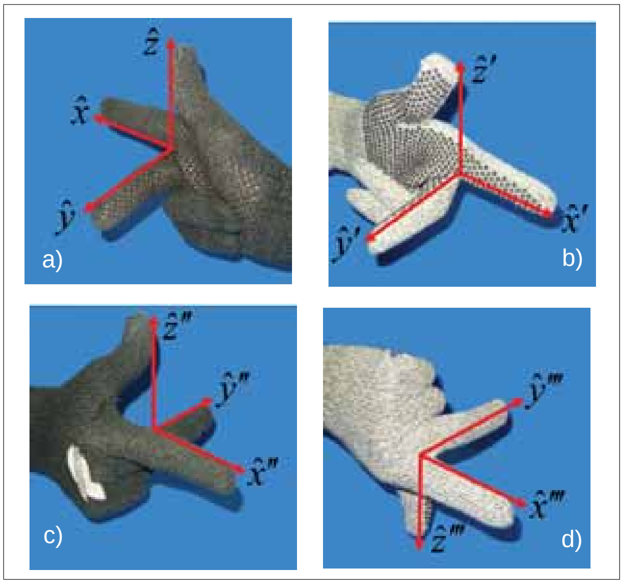

Fig. 2. Three-finger coordinate systems with right hands in dark gloves and left hands in light gloves. (a) A right-handed coordinate system, (b) one axis inverted, (c) two axes inverted, (d) three axes inverted.

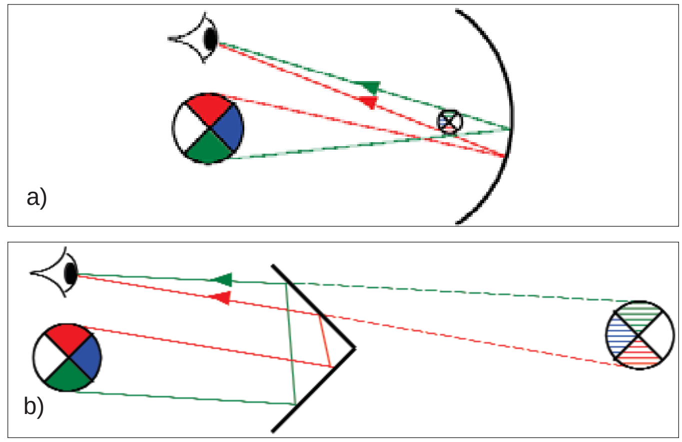

Fig. 3. Image formation by (a) a concave cylindrical with the object outside the focal length and (b) a double mirror.

and for a left-handed coordinate system,

$$\hat{x} \times \hat{y} = -\hat{z}. \qquad (2)$$

Suppose the original coordinates (associated with an object) are right handed as in Fig. 2(a). Inverting one axis (x) corresponds to the coordinate transformation:

$$\hat{x} \to -\hat{x}', \hat{y} \to \hat{y}', \hat{z} \to \hat{z}',$$

so $(-\hat{x}') \times \hat{y}' = \hat{z}'$ or $\hat{x}' \times \hat{y}' = -\hat{z}'$, which is a left-handed system such as in Fig. 2(b).

Other Mirrors

In general, the handedness of an image is related to the number of axes inverted by a mirror. Inverting two axes (x and y) turns the right hand in Fig. 2(a) into the right hand in Fig. 2(c), so handedness is preserved. This corresponds to the coordinate transformation: $\hat{x} \to -\hat{x}''$, $\hat{y} \to -\hat{y}''$, $\hat{z} \to \hat{z}''$, so $(-\hat{x}'') \times (-\hat{y}'') = \hat{x}'' \times \hat{y}'' = +\hat{z}''$, which is right handed. Finally, inverting all three axes turns the right hand in Fig. 2(a) into the left hand in Fig. 2(d), so handedness changes. The coordinate transformation is: $\hat{x} \to -\hat{x}'''$, $\hat{y} \to -\hat{y}'''$, $\hat{z} \to -\hat{z}'''$, so $(-\hat{x}''') \times (-\hat{y}''') = \hat{x}''' \times \hat{y}''' = -\hat{z}'''$, which is left handed. In summary, if an odd (even) number of axes are inverted, handedness is changed (preserved).

This simple rule gives insight into the effects of various mirrors on handedness. All mirrors invert front to back like the plane mirror. Concave cylindrical and spherical mirrors with the object inside the focal length, and convex cylindrical and spherical mirrors with any object placement do not invert any other axis, so they affect handedness in the same way as a plane mirror does. More discussion is required for concave cylindrical and spherical mirrors with the object outside the focal length, and for double and triple mirrors — perpendicular combinations of two and three plane mirrors.[5] Although the last two produce multiple images, we focus on the corner image that is formed by two (three) reflections for a double (triple) mirror.[3]

Figure 3 illustrates image formation by the concave cylindrical mirror with the object outside the focal length, and by the double mirror. Comparing the orientation of the red and green rays in Fig. 3 and in Fig. 1(a) reveals that these mirrors invert two perpendicular axes, front to back (blue to white) and top to bottom (red to green). It is interesting to note that the concave cylindrical mirror achieves this with single reflections of rays, while the double mirror achieves the same axis inversion with pairs of reflections. With their two axes inversions, both of these mirrors preserve handedness.

An alternative explanation of the double mirror is easily extended to the triple mirror, where illustration is difficult. In general, the image from one mirror can

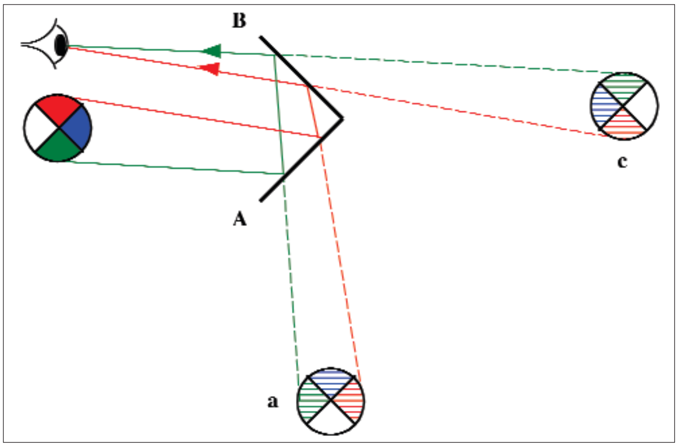

Fig. 4. The object reflects in mirror A to produce image a, which reflects in mirror B to produce corner image c.

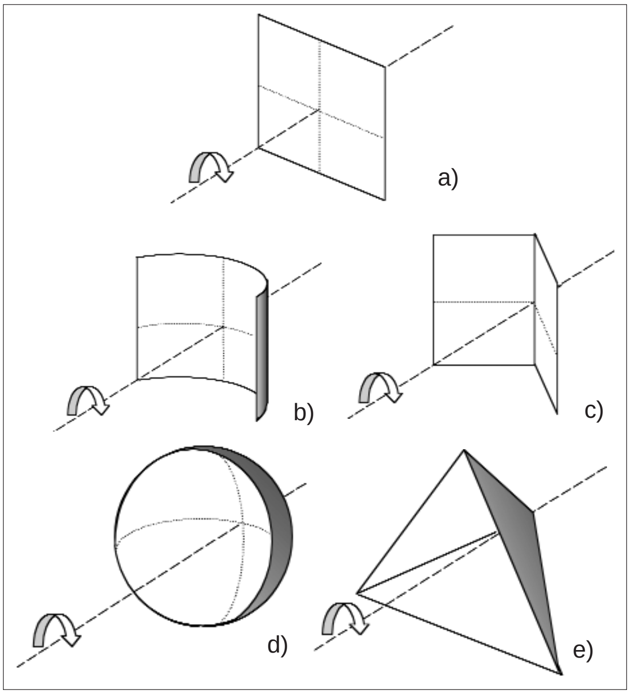

Fig. 5. Axes of rotation for (a) plane, (b) concave cylindrical, (c) double, (d) concave spherical, and (e) triple mirrors.

be considered the object for another.[7] Figure 4 illustrates that when the object is reflected in mirror A to produce image a, the axis perpendicular to A is inverted. Considering image a to be the object for mirror B, the reflection producing corner image c inverts the axis perpendicular to B. The result of the two reflections from the two perpendicular plane mirrors is the inversion of two perpendicular axes. Applying the same reasoning to the triple mirror, since the corner image is produced by a series of three reflections in three perpendicular plane mirrors, it is inverted along three perpendicular axes. The three axis inversions correspond to a change of handedness.

A concave spherical mirror with the object outside the focal length achieves the same result as the triple mirror does, but with single reflections. Any cross section of this mirror inverts the image as in Fig. 3(a). Therefore, in addition to inverting front to back, the mirror inverts the two perpendicular axes along its face. Again, inverting all three axes results in a change of handedness. It should be emphasized that while students may initially identify their images as "upside down" in either this or the triple mirror, they are in fact completely inverted (front to back, left to right, and top to bottom).

Table I summarizes the effects of all of these mirrors on handedness and axis inversions. Pairs of stu-

Table I. Classification of mirrors.

Effects of Mirror	Type of Mirror: Single Reflections	Type of Mirror: Multiple Reflections
• changes handedness • inverts one axis	Plane mirror Convex cylindrical & spherical mirrors Concave cylindrical/spherical with object inside*	
• preserves handedness • inverts two axes	Concave cylindrical mirror with object outside*	Double mirror's corner image
•changes handedness • inverts three axes	Concave spherical mirror with object outside*	Triple mirror's corner image

***** The location of the objects "inside" or "outside" are relative to the focal length.

dents can explore these effects in the following way. One student holds up a right hand, as in Fig. 2(a), in front of a mirror. A second student attempts to reproduce the image hand's orientation with either a left or a right hand. Comparing each other's hands, they can determine the number of axes (fingers) that have their directions reversed and if handedness has changed.

Rotating Mirrors and (Sometimes) Images

A phenomenon closely related to handedness is how an image changes for a fixed object as a mirror is rotated. This can be explained in terms of axis inversions, and in certain cases, simply by the symmetry of the mirror. Figure 5 shows the axes of rotation that will be considered for the various mirrors discussed.

Since the plane mirror does not invert any axis perpendicular to the axis of rotation, an image in it does not change as the mirror is rotated. This also follows from symmetry of the plane mirror about the axis of rotation. Similarly, rotating any other mirror that is symmetric about its rotation axis, such as a spherical mirror, leaves an image unchanged. In the cases of the concave spherical mirror with the object outside the focal length and the triple mirror,[5] all three coordinate axes are inverted. Regardless of how the three coordinate axes are chosen, this results in a complete inversion of the image, so rotating these mirrors does not affect an image's orientation.

The mirrors that invert two axes (and preserve handedness) produce more interesting results when they are rotated, so they make intriguing displays.[4,5,6] The concave cylindrical mirror with the object outside the focal length and the double mirror both invert the axis of rotation and one axis perpendicular to that, so they have the same effect on the image's orientation. Figure 6 shows the result of these two axis inversions with a double mirror in three different orientations. The orientation of the mirror in Fig. 6(a) is the same as in Fig. 5(c). Notice in Figs. 6 (b) and (c) that the image has rotated in the same direction as the mirror, but through twice the angle.[5] Indeed, when the mirror is rotated through 180°, the image rotates through 360° and back to its original orientation, as expected from the symmetry of the mirror.

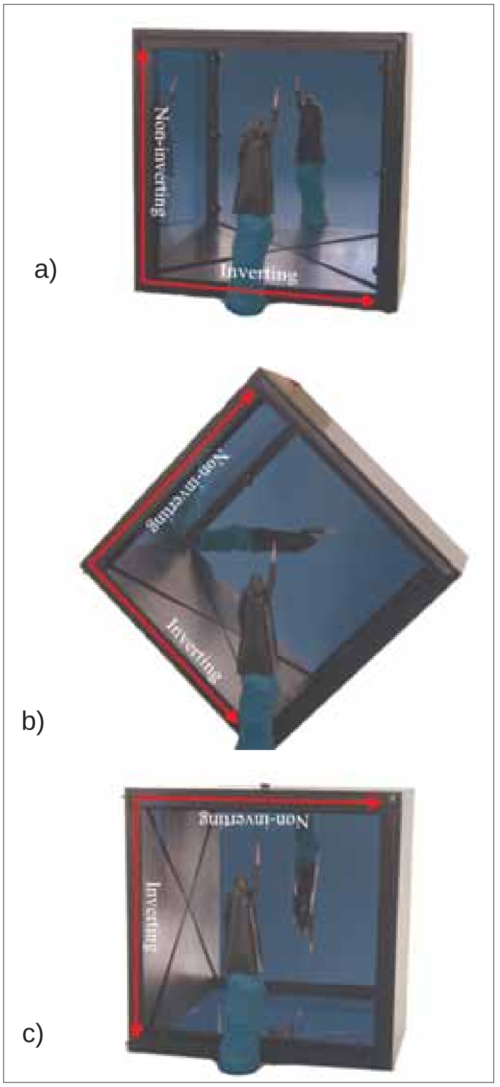

Fig. 6. The change in the image of a fixed object with rotation of the double mirror. The noninverting axis of the mirror is (a) vertical, (b) rotated clockwise by 45°, and (c) horizontal (rotated by 90°).

Since it is the relative orientation of the mirror and an object that is important, students can make observations with a stationary double mirror and a rotating object to understand why an image rotates twice as far as the mirror does relative to an object. A student could start by standing upright in front of a mirror that has its inverting axis horizontal, as in Fig. 6(a). If the student tilts to the right by a certain amount, then the image tilts to the student's left (the image's right) by the same amount because the mirror inverts the horizontal axis. The result is that the angle between the student and the image has changed by twice as far as the student tilted.

Summary

We have presented a systematic treatment of the handedness of images in terms of axis inversions. We have related these inversions to the effects of rotating mirrors on image orientation. Along the way, we have also suggested how these topics can be presented in introductory courses at a purely conceptual level, or with the addition of some simple mathematics.

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7. This approach is similar to the virtual mirror method. See for example: T.B. Greenslade Jr., "Multiple images in plane mirrors," *Phys. Teach.* **20**, 29–33 (Jan. 1982).
8. The photographs were taken using a "True Mirror" (#MIR-300) from Educational Innovations, Inc., 362 Main Ave., Norwalk, CT 06851, 888-912-7474; http://www.teachersource.com. This double mirror is constructed with front-surface mirrors, so the seam is almost invisible.

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