

The Dizzying Depths of the Cylindrical Mirror

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A typical introduction to geometrical optics treats plane and spherical mirrors. At first glance, it may be surprising that texts seldom mention the cylindrical mirror, except for the occasional reference to use in fun houses and to viewing anamorphic art.^{1,2} However, even a cursory treatment reveals its complexity. Holzerlein used an extended object to qualitatively illustrate that images are produced both before and behind a concave cylindrical mirror.³ He also speculated on how this extreme astigmatism results in an observer's dizziness. By considering a simple point object, we make a more detailed analysis of the cylindrical mirror and the dizziness it induces. First, we illustrate how rays from a point object reflect to form not one point image but two line images. Next, we describe how an observer perceives a likeness of the object. Finally, we suggest how confusing depth cues induce dizziness. Although we focus on the concave cylindrical mirror, the discussion is easy to generalize to the convex cylindrical mirror.

Astigmatism and Line Images

The image of a point source is the point through which all of its reflected rays either converge or project. Familiar examples of these two cases are the real image of an object (outside the focal length) reflected in a concave spherical mirror and the virtual image of an object reflected in a plane mirror. A concave cylindrical mirror has a circular cross section like a spherical mirror and a straight cross section like a plane mirror. Figure 1 illustrates how rays reflected from the former converge to a point in front of the mirror,⁴

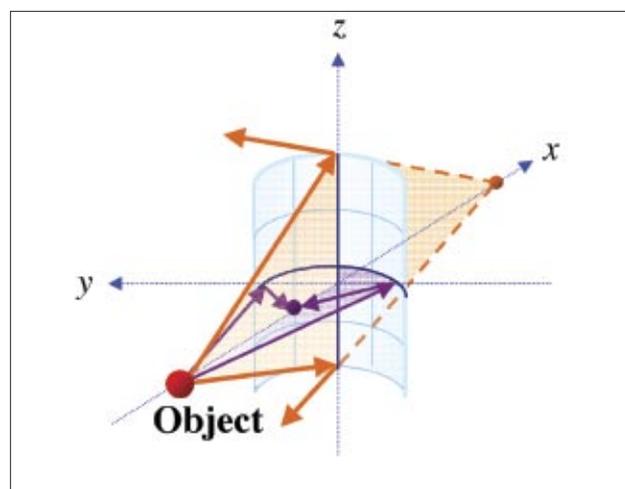


Fig. 1. Reflections from cross sections of a concave cylindrical mirror. Rays reflected by a straight section of the mirror project to a point behind it, and those reflected by a concave section converge to a point in front of it.

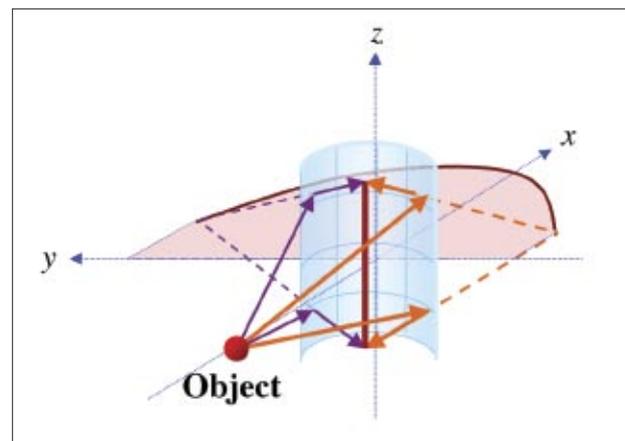


Fig. 2. Two line images are formed when a point object is reflected in a concave cylindrical mirror.

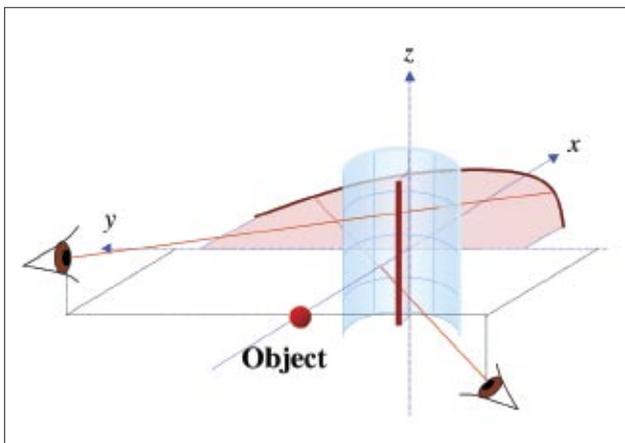


Fig. 3. When viewing a likeness, the ray entering an eye intersects one line image and projects back to the other.

while rays reflected from the latter project to a point behind the mirror.

The picture gets more complicated when we consider the rays reflecting from all points on the cylinder, not just those along narrow sections. Figure 2 shows how the reflected rays converge to points along a line in front of the mirror and project to points along a curve behind the mirror. For example, the two upper reflected rays converge to a point above and in front of the mirror, while the two on the right project back to a point to the right and behind the mirror. Reflections from the whole mirror produce the two line images shown in Fig. 2, which are similar to those produced by an astigmatic lens.⁵ The real line image is straight because of the symmetry of the mirror along the z -axis. We show in an appendix available online⁶ that the virtual line image is a segment of a limaçon.⁷

Depth and Dizziness

Although a point object is mapped to two line images, observers who look at their reflections in a concave cylindrical mirror will recognize their likenesses (despite their being blurred, distorted, and reoriented^{8,9}). In order to understand how the likeness of a point object is perceived, it is important to consider the role of an observer.

For a point object, each ray reflected from the concave cylindrical mirror both intersects the line image in front of the mirror and projects back to the line image behind it. Therefore, the ray connecting these two

lines and the eye determines the direction in which the likeness is viewed. This is shown in Fig. 3 for two eyes in different locations. Unlike an image in which a point of an object is mapped to a single point, the likeness cannot be considered to have a fixed location. As an observer moves left or right, the direction of the gaze changes as if the likeness were located at the line image in front of the mirror. However, as an observer moves up and down, the direction of the gaze changes as if the likeness were located at the line image behind the mirror.

Viewing with two eyes provides similar depth information to that gathered by using one eye and moving around. Because of their physical separation, the eyes look in slightly different directions to view the same object. The nearer the object, the larger the angle between the directions in which the eyes gaze. This “angle of convergence” cues the observer to the object’s depth.¹⁰ For two eyes oriented horizontally (in the x - y plane in Fig. 3) the two rays of their gazes cross at the line image in front of the mirror, indicating that the likeness is there. However, if the eyes are oriented vertically (in the x - z plane), the two rays project to the line image behind the mirror, indicating that the likeness is there. When the two eyes are off axis, their gazes do not converge.

A single, stationary eye also provides a depth cue, the amount of accommodation required to focus.¹⁰ The focal length of the lens is adjusted by the ciliary muscles, so their tension gives information about the distance of an object. If the muscles are relaxed, then the object is far away; if they are tense, the object is nearby. This means of determining depth is complicated by a cylindrical mirror’s two line images. A lens focused to the front line’s depth would project a vertical line segment on the retina, while one focused to the back line’s depth would project a horizontal segment. If the lens were focused at an intermediated distance, an oval shape would be projected on the retina.¹¹ We suspect that the eye compromises and focuses at a point between the line images where each axis is equally well focused. This is called the “circle of least confusion” for an astigmatic lens.⁵ It is our experience that when viewed with just one eye, a reflection in a cylindrical mirror is fuzzy but not dizzying.

Dizziness is likely a result of confusing depth cues. When an observer’s eyes are along either the horizon-

tal or the vertical axis, the angle of convergence places the likeness at the depth of one of the line images, but accommodation places it between the two line images. Identifying the exact physiological cause of the dizziness would require further exploration, but we can conceive of two possibilities. The inability to mentally reconcile these inconsistent depth cues could produce a confusing sensation. Alternatively, the focus and the aim of the eyes may keep adjusting in a futile attempt to achieve both a clear likeness and consistent depth cues, producing an uneasy sensation of motion. If the observer's eyes are not along either axis, convergence does not give a clear depth cue, which seems to make the confusion worse. In our experience, the dizzying effect is the worst while tilting one's head from side to side.

Summary

We have explored the interesting properties of the concave cylindrical mirror by building on the familiar descriptions of plane and spherical mirrors. As with an astigmatic lens, the cylindrical mirror maps a point object to two line images. However, an observer can resolve a recognizable likeness of an object. The different depth cues provided by accommodation and convergence are probably the cause of the dizziness an observer may experience. The analysis for a convex cylindrical mirror is the same, except that both line images are virtual. The depth cues are not as confused for object points near the convex cylindrical mirror since the line images are not very far apart.

References

1. See, for example, Thomas D. Rossing and Christopher J. Chiaverina, *Light Science: Physics and the Visual Arts* (Springer, New York, 1999), pp. 52–53 and pp. 60–62.
2. McLoughlin Bros., *The Magic Mirror: An Antique Optical Toy* (Dover Publications, New York, 1979). This book of anamorphic pictures comes with a flexible mir-

ror that can be formed into a cylinder.

3. Thomas M. Holzberlein, "How to become dizzy with Derman's optical puzzle," *Phys. Teach.* **20**, 401–402 (Sept. 1982).
4. Since the cross section is circular, not parabolic, this only holds for paraxial rays.
5. Frank J. Pedrotti and Leno S. Pedrotti, *Introduction to Optics*, 2nd ed. (Prentice Hall, Upper Saddle River, NJ, 1993), pp. 98–100.
6. See EPAPS Document E-PHTEAH-43-012502. This document may be retrieved via the EPAPS homepage (<http://www.aip.org/pubservs/epaps.html>) or from [ftp.aip.org](ftp://ftp.aip.org) in the directory/epaps/ in the phys_teach folder. See the EPAPS homepage for more information.
7. E.H. Lockwood, *A Book of Curves* (Cambridge University Press, Cambridge, 1967), pp. 45–51.
8. Samuel Derman, "An optical puzzle that will make your head spin," *Phys. Teach.* **19**, 395 (Sept. 1981).
9. Alan J. DeWeerd and S. Eric Hill, "Reflections on handedness," *Phys. Teach.* **42**, 275–279 (May 2004).
10. David Falk, Dieter Brill, and David Stork, *Seeing the Light: Optics in Nature, Photography, Color, Vision, and Holography* (Wiley, New York, 1986), pp. 207–209.
11. This can be demonstrated by modeling the lens and retina with a converging lens in front of a sheet of paper. A small light bulb shielded so that it only shines toward the cylindrical mirror approximates a point object.

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Appendix to “The Dizzying Depths of the Cylindrical Mirror”

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Here, we show that the virtual line image produced by a cylindrical mirror for a point source is a limaçon (or limaçon of Pascal). All rays from a point object, incident along the same vertical strip of a cylindrical mirror (e.g., the purple rays in Fig. 2), project to one point. If there were a plane mirror running tangent to the strip, this point would be the image. The point lies as far behind the tangent plane mirror as the object lies in front of it. A view of this in the x - y plane (with the object at the origin) is shown in Fig. 4, where \mathbf{B} is the ray from the object to a point on the cylindrical mirror, ρ is the vector from the object to the corresponding point on the line image, \mathbf{R} is the radius of the mirror that points to where the ray reflects, and \mathbf{C} extends from the object to the center of the mirror. Recognizing some similar angles and using the law of cosines leads to the following relations:

$$\frac{1}{2} \rho = B \cos(\theta - \phi) \quad (1)$$

$$B^2 = C^2 + R^2 + 2CR \cos \theta, \quad (2)$$

and

$$C^2 = R^2 + B^2 - 2RB \cos(\theta - \phi). \quad (3)$$

Simplifying these equations yields

$$\rho = 2R + 2C \cos \theta, \quad (4)$$

which is the polar equation for a limaçon.⁷ Figure 5 shows the limaçon produced when a point object is reflected by a cylinder. Reflection from the concave (blue) portion of the mirror produces the outer (blue) loop of the limaçon, while reflection from the convex (red) portion produces the inner (red) loop.

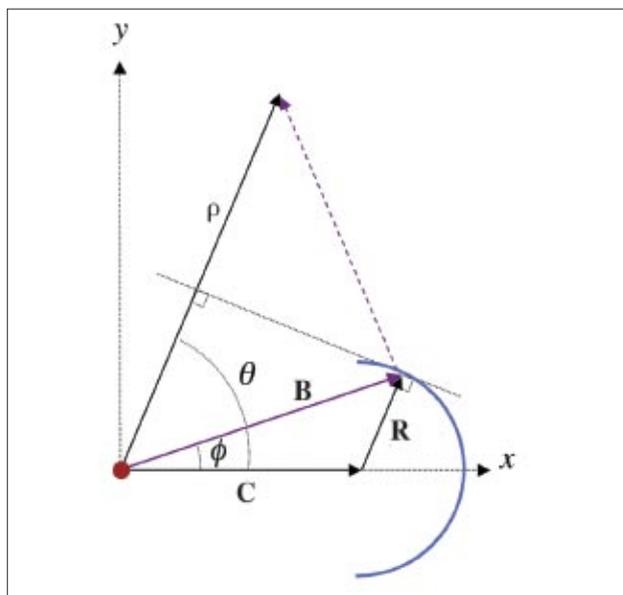


Fig. 4. Construction for finding a point on the virtual line image.

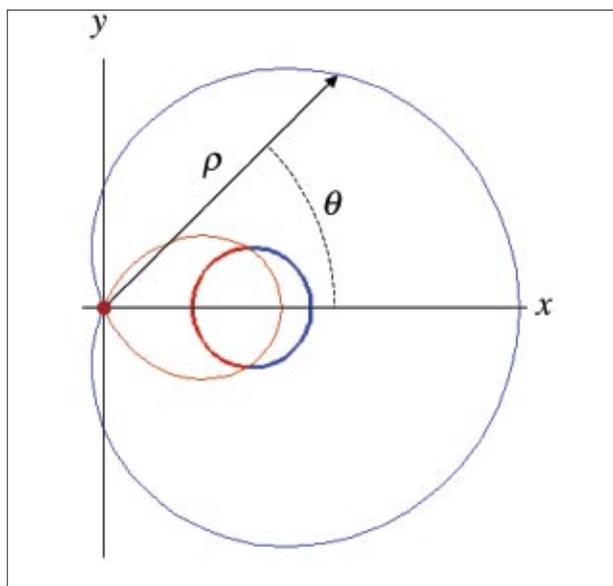


Fig. 5. Virtual line image of a point object reflected from a cylinder is a limaçon. Reflections from the blue and red segments of the cylinder produce the blue and red loops, respectively.